

Simple New Axioms for Quantum Mechanics

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February 1, 2008

Abstract

The space \mathcal{P} of pure states of any physical system, classical or quantum, is identified as a Poisson space with a transition probability. The latter is a function $p : \mathcal{P} \times \mathcal{P} \rightarrow [0, 1]$; in addition, a Poisson bracket is defined for functions on \mathcal{P} . These two structures are connected through unitarity. Classical and quantum mechanics are each characterized by a simple axiom on the transition probability p . Unitarity then determines the Poisson bracket of quantum mechanics up to a multiplicative constant (identified with Planck's constant). Superselection rules are naturally incorporated.

1 Introduction

Axiomatic quantum mechanics (cf. [1, 2] for representative overviews) is usually inspired by a mixture of two extreme attitudes. On the one hand, one could try to show that the Laws of Thought necessarily imply that Nature has to be described by quantum mechanics. On the other hand, quantum mechanics could be a contingent theory. In this Letter I will show that quantum mechanics can be described by one axiom that is fairly general, incorporates classical mechanics as well, and may fall into the first category, and by two further axioms which, in my opinion, are clearly contingent.

The purpose of my axiomatization is twofold. Firstly, it suggests at what point quantum mechanics may be modified. Secondly, it formalizes classical and quantum mechanics in parallel, so that it becomes crystal-clear to what extent these two theories agree, and where they (dramatically) differ. Thus I expect the structure set out below to be useful in the theory of quantization as well as of the classical limit of quantum mechanics.

The usual formulation of quantum mechanics in terms of linear operators on a Hilbert space obviously does not serve this second purpose well. However, even more refined schemes involving convex structures and/or quantum logic [1, 2] are of limited use in the present context: apart from their (arguable) lack of intuitive appeal, such approaches miss the Poisson structure that plays such a crucial role in mechanics [3]. Nonetheless, the mathematical proofs that are necessary to show that the axioms proposed below indeed describe quantum mechanics use these schemes in an essential way.

2 Poisson brackets

The pure states of a classical mechanical system are the points of its phase space \mathcal{P} . This space is equipped with a Poisson structure, that is, for any two (smooth) functions f, g on \mathcal{P} the Poisson bracket $\{f, g\}$ is defined. One calls \mathcal{P} a *Poisson manifold*. Thus any (smooth) function H on \mathcal{P}

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defines a *Hamiltonian vector field* X_H on \mathcal{P} by $X_H(g) = \{H, g\}$, and the Hamiltonian equations of motion satisfied by a curve $\sigma(t)$ in \mathcal{P} are [3]

$$\frac{d\sigma(t)}{dt} = X_H(\sigma(t)). \quad (1)$$

In the sixties it was discovered by many people (cf. [3] for a modern presentation and references) that quantum mechanics may, to some extent, be brought into the same form. Here one chooses $\mathcal{P} = \mathbf{P}(\mathcal{H})$, the projective space of \mathcal{H} , the Hilbert space of (pure) states of the system. Every Hermitian linear operator \hat{A} on \mathcal{H} defines a real-valued function f_A on \mathcal{P} by

$$f_A(\psi) = \langle \psi | \hat{A} | \psi \rangle / \langle \psi | \psi \rangle, \quad (2)$$

where $\psi \in \mathbf{P}(\mathcal{H})$ is the image of $|\psi\rangle \in \mathcal{H}$. The Poisson bracket of such functions is essentially given by the commutator:

$$\{f_A, f_B\} = \frac{i}{\hbar} f_{[\hat{A}, \hat{B}]}. \quad (3)$$

The Schrödinger equation (projected to $\mathbf{P}(\mathcal{H})$) is then precisely (1), with $H = f_H$.

Hence quantum mechanics may be described in the language of classical mechanics, with some curious extra rules: the phase space is $\mathcal{P} = \mathbf{P}(\mathcal{H})$, the Poisson bracket is defined by (3), and only functions of the form f_A (rather than all smooth functions on \mathcal{P} , as in classical mechanics) correspond to observables (but note that the Poisson bracket of any two functions on \mathcal{P} is determined by the special case (3)).

In order to formulate the axioms below, I need to recall a basic theorem on Poisson structures [3]. Namely, every Poisson manifold \mathcal{P} can be decomposed as the union of its *symplectic leaves*: these are maximal subspaces on which the Poisson structure is non-degenerate. This means that at each point ρ the Hamiltonian vector fields X_f span the tangent space of the leaf through ρ .

3 Transition probabilities

It was known from the earliest days of quantum mechanics that the notion of a transition probability is of central importance to this theory. Abstractly, a transition probability p on a set \mathcal{P} is a function on $\mathcal{P} \times \mathcal{P}$, taking values in the interval $[0, 1]$, with the special property that $p(\rho, \sigma) = 1$ is equivalent to $\rho = \sigma$ [4, 2]. Moreover, in general one assumes the symmetry property $p(\rho, \sigma) = p(\sigma, \rho)$. In standard quantum mechanics one puts $\mathcal{P} = \mathbf{P}(\mathcal{H})$, as above, and

$$p(\rho, \sigma) = |\langle R | \Sigma \rangle|^2, \quad (4)$$

where $|R\rangle$ and $|\Sigma\rangle$ are unit vectors in \mathcal{H} which project to ρ and σ in $\mathbf{P}(\mathcal{H})$, respectively. This choice of p is essentially the Born rule. Note that $p(\rho, \sigma) = f_{[\sigma]}(\rho)$, where the linear operator $[\sigma]$ on \mathcal{H} is the orthogonal projector onto the one-dimensional subspace spanned by $|\Sigma\rangle$.

The physical meaning of transition probabilities implies that in the case of classical mechanics, where \mathcal{P} is an arbitrary manifold, one has to put

$$p(\rho, \sigma) = \delta_{\rho\sigma} \quad \forall \rho, \sigma. \quad (5)$$

In what follows I need the (obvious) result that any space with a transition probability decomposes as the union of its irreducible components, called *sectors* (a subspace is irreducible if it is not the union of two mutually orthogonal spaces). Also, there are some technical requirements on transition probabilities (each maximal orthogonal subset being a basis, and orthoclosed subsets being subspaces, cf. [2] and [5]) that I shall assume.

The superposition principle of quantum mechanics (which is normally expressed in terms of vectors in a Hilbert space) can be described in the present language. For any subset Q of \mathcal{P} one defines the orthoplement

$$Q^\perp = \{\sigma \in \mathcal{P} | p(\rho, \sigma) = 0 \quad \forall \rho \in Q\}. \quad (6)$$

The possible superpositions of the pure states ρ, σ are then the elements of $\{\rho, \sigma\}^{\perp\perp}$. If ρ and σ lie in different sectors then clearly $\{\rho, \sigma\}^{\perp\perp} = \{\rho, \sigma\}$.

4 The axioms

The mathematical structure characterizing pure state spaces in classical and quantum mechanics can now be identified. A *Poisson space with a transition probability* is at the same time a transition probability space (\mathcal{P}, p) and a Poisson manifold $(\mathcal{P}, \{, \})$, such that the Poisson structure is *unitary* in the following sense. For each $\rho \in \mathcal{P}$ I define a function p_ρ on \mathcal{P} by $p_\rho(\sigma) = p(\rho, \sigma)$. Using p_ρ as the Hamiltonian H , the Hamiltonian flow $\sigma(t)$ on \mathcal{P} is given by the solution of (1). Unitarity now means that for each ρ this flow leaves the transition probabilities invariant, in that $p(\sigma_1(t), \sigma_2(t)) = p(\sigma_1, \sigma_2)$ for all $\sigma_1, \sigma_2 \in \mathcal{P}$ and all t .

My axioms on the pure state space \mathcal{P} of quantum mechanics with (discrete) superselection rules are:

- QM1: The pure state space \mathcal{P} is a Poisson space with a transition probability;
- QM2: For each pair (ρ, σ) of points which lie in the same sector of \mathcal{P} , $\{\rho, \sigma\}^{\perp\perp}$ is isomorphic to $P(\mathbf{C}^2)$ as a transition probability space;
- QM3: The sectors of (\mathcal{P}, p) coincide with the symplectic leaves of $(\mathcal{P}, \{, \})$.

Here $P(\mathbf{C}^2)$ is understood to be equipped with the usual Hilbert space transition probabilities. Axiom QM2 was inspired by a mathematical paper on convexity theory and operator algebras [6].

To characterize classical mechanics, one simply postulates the axioms CM1 = QM1, and CM2 = eq. (5). In this case, $\{\rho, \sigma\}^{\perp\perp}$ simply equals $\{\rho, \sigma\}$, and each point is a sector.

5 Consequences of the axioms

As outlined in the next section, it can be shown that the axioms QM1-QM3 imply that $\mathcal{P} = \cup_i P(\mathcal{H})_i$ (which is meant as a union over sectors). Here each \mathcal{H}_i is a Hilbert space, and the transition probabilities in each sector $P(\mathcal{H})_i$ are given by (4). Moreover, the Poisson bracket on \mathcal{P} is determined up to a multiplicative constant \hbar , and is such that in each sector (or, equivalently, symplectic leaf) $P(\mathcal{H})_i$ it is given by (3). I find it satisfying to see Planck's constant enter as a free parameter allowed by the axioms.

This illustrates a remarkable difference between classical and quantum mechanics: in the former the pure state space and the Poisson structure can be freely specified, whereas in the latter the only freedom lies in the dimensions of the \mathcal{H}_i and the (nonzero) value of \hbar .

I now show how the usual observables of quantum mechanics can be reconstructed, restricting myself to the finite-dimensional case (see [5] for the general construction, which also applies to classical mechanics). Firstly, the space of observables is simply the real vector space $\mathcal{A}(\mathcal{P})$ of (real) finite linear combinations of the functions p_ρ , where ρ runs through \mathcal{P} . In other words, the observables of quantum mechanics are in essence the transition probabilities. Secondly, one has a spectral theorem in $\mathcal{A}(\mathcal{P})$: every function $f = \sum \mu_i p_{\rho_i}$ can be rewritten as $f = \sum_j \lambda_j p_{e_j}$, where $p(e_j, e_k) = \delta_{jk}$. This gives us a squaring map $f^2 := \sum_j \lambda_j^2 p_{e_j}$, and subsequently a commutative product (Jordan structure) by

$$f \circ g = \frac{1}{4}((f + g)^2 - (f - g)^2), \quad (7)$$

which is bilinear because of the special form (4) of p . I now complexify $\mathcal{A}(\mathcal{P})$, and define a product \cdot on $\mathcal{A}(\mathcal{P})_{\mathbf{C}}$ by

$$f \cdot g = f \circ g - \frac{1}{2}i\hbar\{f, g\}. \quad (8)$$

This product turns out to be associative as a consequence of the unitarity relating the transition probability (which is ultimately responsible for the product \circ) and the Poisson bracket. Finally, one (easily) shows that the algebra $(\mathcal{A}(\mathcal{P})_{\mathbf{C}}, \cdot)$ thus constructed is a direct sum of matrix algebras. Indeed, in case of a single sector the usual spectral theorem already says that any function f_A lies in $\mathcal{A}(\mathcal{P})$ (for Hermitian \hat{A}). In any case, it is pleasant to represent observables as real-valued functions on the space of pure states, just like in classical mechanics.

6 Outline of the proof

I here give the main steps in showing that the axioms imply that $\mathcal{P} = \cup_i \mathcal{P}(\mathcal{H})_i$, with Poisson structure as indicated. For full details see [5]; lattice-theoretic definitions may be found in [7] or [2]. For simplicity I restrict myself here to the irreducible case of one sector.

Firstly, on the basis of Axiom QM1 one constructs a complete atomic orthomodular lattice $\mathcal{L}(\mathcal{P})$ [2], whose members are the orthoclosed subspaces of \mathcal{P} . Axiom QM2 then leads to the conclusion that this lattice has the covering property (i.e., satisfies the exchange axiom); the proof is by induction on the dimension of the lattice. One then uses the (von Neumann) coordinatization procedure for projective lattices [2], which leaves one with an arbitrary division ring \mathbf{D} . *This step can only be performed if the dimension of \mathcal{P} (as a transition probability space) is not equal to 3, and this limitation restricts my result.* One then uses Axiom QM2 once again to prove that $\mathbf{D} = \mathbf{C}$; surprisingly, this is the most difficult step. Standard theorems [2] then lead to the conclusion that the transition probability must come from the usual inner product on a complex Hilbert space \mathcal{H} . Hence $\mathcal{P} = \mathcal{P}(\mathcal{H})$.

I now come to the identification of the Poisson structure on \mathcal{P} . Axiom QM3 implies that each sector is a symplectic space. Unitarity (in my sense) and Wigner's theorem (cf. [2]) imply that each f_A generates a flow on $\mathcal{P}(\mathcal{H})$ which is the projection of a unitary flow on \mathcal{H} . Therefore, $\{f_A, f_B\}(\sigma) = \frac{d}{dt} f_B(\exp(itC(A))\sigma)_{t=0}$ for some Hermitian operator \hat{C} on \mathcal{H} , depending on A (here $\exp(itC(A))\sigma$ is the projection of $\exp(it\hat{C}(A))|\Sigma\rangle$, using my previous notation). The right-hand side equals $f_{i[\hat{C}, \hat{B}]}$. Anti-symmetry of the left-hand side implies that $\hat{C} = \hbar \hat{A}$ for some $\hbar \in \mathbf{R}$, and this leads to the desired result. The multiplicative constant \hbar must be nonzero in order to satisfy Axiom QM3. In principle, it may depend on the sector, but given that it is nonzero this can be undone by a simple rescaling of the Poisson bracket.

7 Beyond quantum mechanics

In my opinion, the most remarkable aspect of these axioms lies in the universality of the transition probabilities in quantum mechanics. Take any quantum system, and any two of its pure states: axiom QM2 describes their superpositions and transition probabilities. This strongly suggests that there should be some underlying explanation for these transition probabilities. The central limit theorem of probability theory comes to mind: whatever the individual probability distribution, if one has a large number of replicas one will find that fluctuations are described by the Gaussian (normal) distribution. One would hope that the 'distribution' (4) emerges in a similar way as some universal limit.

References

- [1] S.P. Gudder, *Stochastic Methods in Quantum Mechanics* (North-Holland, Amsterdam, 1979).
- [2] E.G. Beltrametti and G. Cassinelli, *The Logic of Quantum Mechanics* (Cambridge University Press, Cambridge, 1984).
- [3] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry* (Springer, New York, 1994).
- [4] J. von Neumann, *Continuous geometries with a transition probability*, Mem. Amer. Math. Soc. **252** (1981) (MS from 1937).
- [5] N.P. Landsman, quant-ph/9603005 (submitted to Rev. Math. Phys.).
- [6] E.M. Alfsen, H. Hanche-Olsen, and F.W. Shultz, Acta Math. **144**, 267 (1980).
- [7] G. Kalmbach, *Orthomodular Lattices* (Academic Press, London, 1983).